Journal of Engineering Mathematics, Vol. 8, No. 4, October 1974
Noordhoff International Publishing – Leyden
Printed in The Netherlands

A plane wave expansion theorem for cylindrically radiated fields

S. N. KARP*

Courant Institute of Mathematical Sciences, New York University, New York, USA

N. R. ZITRON**

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907, USA
(Received February 27, 1974)

SUMMARY
An asymptotic expansion for two-dimensional outwardly radiating fields is developed from an integral representation of these fields by means of a saddle point integration. The expansion is given in terms of inverse powers of the distance from a point in a fixed region to a point in a circular neighborhood at a large distance from that region. The coefficients are expressed in terms of plane waves and linear combinations of derivatives of plane waves with respect to angle of incidence. The theorem may be employed in scattering problems in reducing scattering of arbitrary two-dimensional fields by arbitrary cylinders to scattering of plane waves by the arbitrary cylinders.

1. Introduction

Plane waves and cylindrical waves are related to each other. In fact, the radiation from a collection of cylindrical sources within a bounded cylindrical domain, when viewed by an observer at an infinite distance, cannot be distinguished from a plane wave. As the observer approaches closer to the source region, the nature of the radiation may appear cylindrical or perhaps even more complicated. If, however, the distance from the source is sufficiently large, it may be said that the wave is almost, but not quite, a plane wave.

This paper is concerned with a particular relationship between a general cylindrically radiated wave and a plane wave. A radiated wave or radiated field is defined as a solution of the Helmholtz equation which satisfies the Sommerfeld radiation condition for an outgoing wave. It is shown here that a radiated wave is expressible as an asymptotic series involving plane waves and their derivatives of various orders with respect to angle, thus furnishing a correction to the plane wave approximation. A few terms of this series, obtained previously by Zitron and Karp [12], were utilized to good advantage in dealing with multiple scattering of plane waves by two widely spaced parallel cylinders of arbitrary shape. The present paper gives an explicit expression for the general term of the series whose coefficients are linear combinations of plane waves and their derivatives with respect to angle of incidence. This representation also provides a more efficient way of calculating the terms of the expansion than was used before (see Zitron and Karp [12]). These results were reported earlier in abstracted form (Karp and Zitron [4]).

The major result presented in this paper is an asymptotic expansion of an integral representation for a cylindrically radiated field in a distant finite region. Application of the saddle point method to this integral yields a power series in inverse half-integral powers of kd, where k is the propagation constant and d is a long distance by which the origin O of a coordinate system located in the source region has been translated. Since the coefficients in the asymptotic expansion turn out to be linear combinations of plane waves and their derivatives of various orders with respect to angle of incidence, the expansion can be used to represent the given field in a new coordinate system as a plane wave plus a correction for the curvature of the wavefront in the neighborhood of a scatterer located near the origin O′ of the new coordinate system. The correction terms appear in the form of derivatives of plane waves with respect to angle of

* Research supported by contract AF19(628)3868 with Air Force Cambridge Research Laboratories.
** The initial research was carried out while one of the authors (N. R. Zitron) was at Harvard University.

incidence. The latter is merely a parameter. Thus the theorem has special utility in scattering problems, since the response to the incident wave can be given in terms of derivatives of the response to a plane wave as was done by the authors (Zitron and Karp [12]) in a previous paper. It is possible, then, by means of this expansion, to determine the scattered field when radiation from a cylindrical region is incident upon a cylindrical scatterer and also to reduce multiple scattering problems to single scattering problems. Both of these cases may be treated in a quite general manner without reference to the detailed nature of the incident radiation or the shape or composition of the scatterer. In this respect, the expansion theorem may be used in place of addition theorems which are applicable only when suited to the special geometry of the sources and scatterers and which, even in such cases, may be clumsier to apply.

2. Derivation of the expansion

Let A be a fixed region of arbitrary size (Figure 1) and let d be the distance from a point O at the center of the smallest circle containing A to the center O′ of a circle B. Let the diameters of the circles A and B be small in comparison with d. Let P be a point in B other than O′. Let r and θ be the polar coordinates of P with origin at O, where θ is measured from the O–O′ axis.

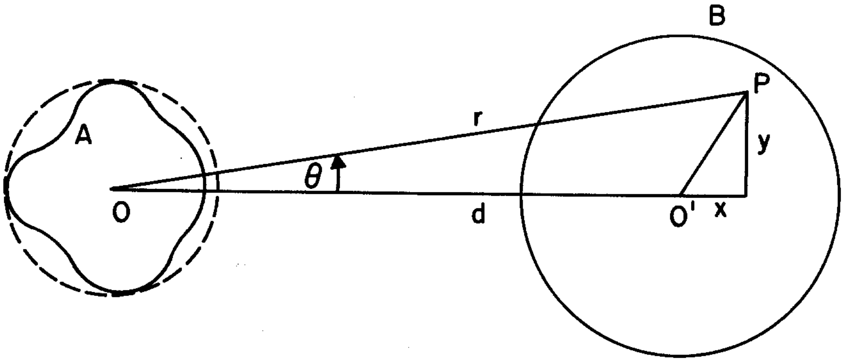

Figure 1. Region A contains a distribution of sources and scatterers. The broken line is a circle circumscribed about A. Region B is a circular neighborhood in which the plane wave expansion is valid.

Let x and y be cartesian coordinates of P with origin at O′. The principal result of this paper may then be stated as follows:

Theorem. *Consider any two-dimensional radiated field represented by*

$$u(r, \theta) = \int_{C_1} f(\beta) e^{ikr\cos(\theta-\beta)} d\beta \tag{1}$$

in the exterior of a bounded domain A with center O containing any combination of sources and scatterers where C_1 is the Sommerfeld contour for $H_0^{(1)}(kr)$ and $f(\beta)$ is analytic. Then $u(r, \theta)$ has an asymptotic expansion in any neighborhood B of a point O′ such that B is disjoint with respect to the smallest circle containing A provided that $kd \gg 1$ where d is the distance between O and O′ (Figure 1). The expansion is given in terms of plane waves and their derivatives with respect to angle as follows:

$$u \sim e^{ikd} \sum_{t=0}^{n} \frac{h^{(2t)}(0)}{(2t)!} \frac{\Gamma(t+\frac{1}{2})}{(kd)^{t+\frac{1}{2}}} \quad \textit{for any } n, \tag{2}$$

where

$$h^{(2t)}(0) = \sum_{p=0}^{2t} \sum_{\lambda=0}^{2t-p} \sum_{j=0}^{\lambda} \frac{C_p^{2t} C_j^{\lambda}}{\lambda!} (\beta^{\lambda}(0))^{(2t-p)} (\beta_s(0))^{(p)} f^{[\lambda-j]}(0) v^{[j]}(0), \tag{2a}$$

and

$\Gamma(t)$ is the Gamma function, $C_j^\lambda = \lambda!/(j!(\lambda-j)!)$, $v(\beta) = e^{ik(x\cos\beta + y\sin\beta)}$,

$$f^{[\lambda-j]}(0) = \left[\frac{d^{\lambda-j}}{d\beta^{\lambda-j}} f(\beta)\right]_{\beta=0} \quad (\beta_s(0))^{(p)} = \left[\frac{d^{p+1}}{ds^{p+1}} \beta(s)\right]_{s=0},$$

$$(\beta^\lambda(0))^{(2t-p)} = \left[\frac{d^{2t-p}}{ds^{2t-p}} \beta^\lambda(s)\right]_{s=0},$$

$$\beta(s) = \arccos(1+is^2),\ \beta_s(0) = \left.\frac{d\beta}{ds}\right|_{s=0}.$$

Proof.
Let

$$u(r,\theta) = \int_{C_1} e^{ikr\cos(\theta-\beta)} f(\beta)\, d\beta$$

(see Stratton [7]) where $f(\beta)$ is analytic and C_1 is the Sommerfeld contour employed in the integral representation of $H_0^{(1)}(kr)$ (See Figure 2). [The time dependence is assumed to be $e^{-i\omega t}$.] Expansion of $\cos(\theta-\beta)$ in the exponent of (1) in terms of x, y, and d (Figure 1) yields

$$u(r,\theta) = \int_{C_1} e^{ik(x\cos\beta + y\sin\beta)} f(\beta) e^{ikd\cos\beta}\, d\beta = \int_{C_1} g(\beta) e^{ikd\cos\beta}\, d\beta, \tag{3}$$

where $g(\beta) = e^{ik(x\cos\beta+y\sin\beta)} f(\beta)$.

It is clear that $e^{ikd\cos\beta}$ varies rapidly for large d and that the saddle point method is appropriate for evaluation of the integral. No difficulty is encountered in shifting the contour as d becomes

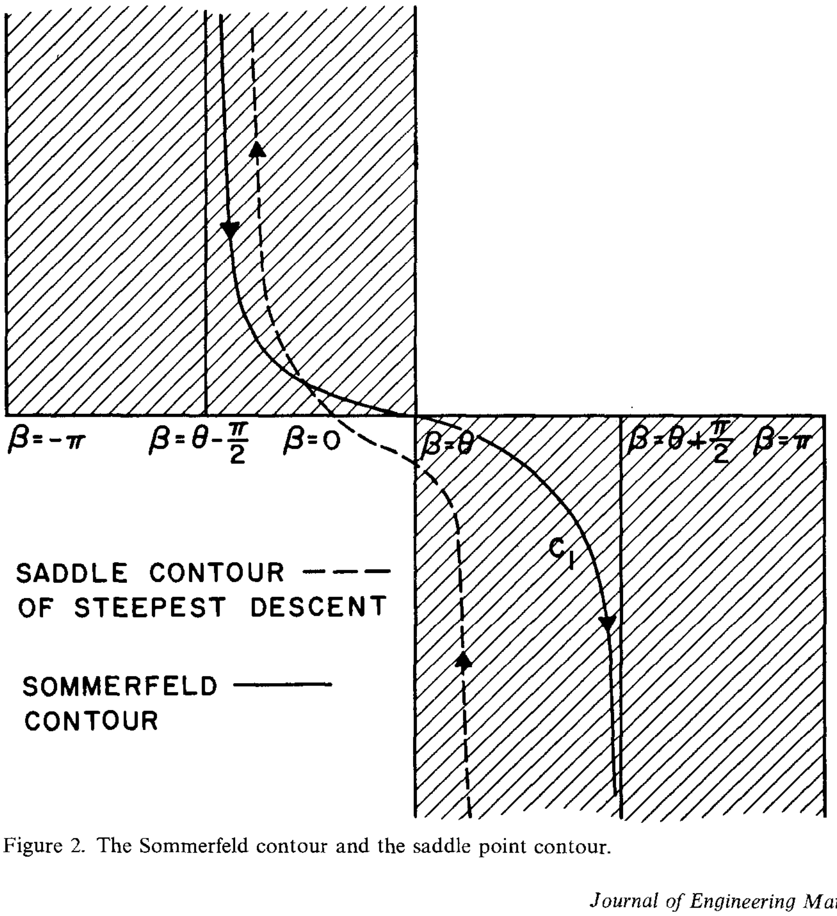

Figure 2. The Sommerfeld contour and the saddle point contour.

large since the integrand is analytic. The transformation $\cos\beta = 1 + is^2$ transforms C_1 into a steepest descent contour (Figure 2) and (3) into

$$u = e^{ikd}\int_{-\infty}^{\infty} g(\beta(s))\frac{d\beta}{ds}e^{-kds^2}ds = e^{ikd}\int_{-\infty}^{\infty} h[\beta(s)]e^{-kds^2}ds\,, \tag{4}$$

where

$$h[\beta(s)] \equiv e^{ik(x\cos\beta(s)+y\sin\beta(s))}f(\beta(s))\ \frac{d\beta}{ds}\ .$$

If $h[\beta(s)]$ is expanded in a Taylor series about the saddle point $s=0$ and a finite number of these terms are retained, it is clear that

$$u \sim e^{ikd}\sum_{m=0}^{n}\left[\frac{h^{(m)}(0)}{m!}\int_{-\infty}^{\infty} s^m e^{-kds^2}ds\right] = e^{ikd}\sum_{m=0}^{n}\frac{h^{(m)}(0)}{m!}I_m\,, \tag{5}$$

where

$$I_m = \int_{-\infty}^{\infty} s^m e^{-kds^2}ds$$

and the superscribed parenthesis signifies differentiation with respect to s.

The function $h^{(m)}(s)$ can be represented in terms of derivatives of plane waves as follows:

$$h^{(m)} = (g\beta_s)^{(m)} = \sum_{p=0}^{m} C_p^m g^{(m-p)}\beta_s^{(p)}$$

by Leibnitz's Rule (Courant [2]).

A rule given by Faa di Bruno (Riordan [5]) for higher order derivatives of implicit functions yields

$$g^{(m-p)} = \sum_{\lambda=0}^{m-p}\sum_{l=0}^{\lambda}(-1)^l\frac{C_l^\lambda}{\lambda!}\beta^l(\beta^{\lambda-l})^{(m-p)}g^{[\lambda]}\,,$$

where the square bracketed superscript signifies differentiation with respect to β. But $\beta(0)=0$. Therefore,

$$g^{(m-p)}(0) = \sum_{\lambda=0}^{m-p}\frac{(\beta^\lambda(0))^{(m-p)}}{\lambda!}g^{[\lambda]}(0)\,.$$

A further application of Leibnitz's rule yields

$$g^{[\lambda]}(0) = \sum_{j=0}^{\lambda} C_j^\lambda f^{[\lambda-j]}(0)v^{[j]}(0)\,.$$

Therefore

$$h^{(m)}(0) = \sum_{p=0}^{m}\sum_{\lambda=0}^{m-p}\sum_{j=0}^{\lambda}\frac{C_p^m C_j^\lambda}{\lambda!}(\beta^\lambda(0))^{(m-p)}(\beta_s^{(p)}(0))f^{[\lambda-j]}(0)v^{[j]}(0)\,.$$

Evaluation of I_m results in

$$I_m = \begin{cases} 0\,, & m \text{ odd} \\ \Gamma(\tfrac{1}{2}m+\tfrac{1}{2})/(kd)^{\frac{1}{2}(m+1)}\,, & m \text{ even} \end{cases}.$$

Let $m=2t$. Then

$$I_{2t} = \begin{matrix} 0\,, & m \text{ odd} & \\ \Gamma(t+\tfrac{1}{2})/(kd)^{\frac{1}{2}(2t+1)}\,, & m \text{ even} & (t=0,1,2,\ldots) \end{matrix}.$$

and

$$u \sim e^{ikd}\sum_{t=0}^{n}\frac{h^{(2t)}(0)}{(2t)!}\frac{\Gamma(t+\frac{1}{2})}{(kd)^{t+\frac{1}{2}}}\,, \tag{6}$$

where

$$h^{(2t)}(0) = \sum_{p=0}^{2t} \sum_{\lambda=0}^{2t-p} \sum_{j=0}^{\lambda} \frac{C_p^{2t} C_j^{\lambda}}{\lambda!} (\beta^{\lambda}(0))^{(2t-p)} (\beta_s(0))^{(p)} f^{[\lambda-j]}(0)\, v^{[j]}(0)\,. \tag{7}$$

It is convenient to note that

$$\beta_s^{(p)}(0) = \begin{cases} 0, & p \text{ odd} \\ 2^{\frac{1}{2}(p+1)} e^{-i[\frac{1}{2}(p+1)]\frac{1}{2}\pi} \{p!/[2^p(\frac{1}{2}p)!]\}^2 = 2^{\nu+\frac{1}{2}} e^{-i(\nu+\frac{1}{2})\frac{1}{2}\pi} [(2\nu)!/(2^{2\nu}\nu!)]^2, & p \text{ even} \end{cases} \tag{8}$$

where $p=2\nu$, $\nu=0, 1, 2, \ldots$.

3. Comparison with a previous expansion of the authors

In a previous paper Zitron and Karp [12] obtained a few terms of an asymptotic series in inverse powers of d by direct multiplication of several Taylor series which could be replaced by derivatives of plane waves with respect to angle. This relationship might have appeared to be a fortuitous accident. In contrast to the previous expansion, the present expansion (6–7) shows that this representation can be obtained systematically by a direct calculation, and is independent of the previous observation. Furthermore, the present expansion gives the general term of the series explicitly.

The calculation of terms in formula (6) is simpler than the previous method of Zitron and Karp [12]. The first two terms in the present expansion, when computed explicitly, agree with the corresponding terms in the previous paper, thus providing a verification. It should be noted that the definition of the complex scattering amplitude $f(\beta)$ of the far field here differs from that of Zitron and Karp [12]. Let $\tilde{f}$ denote the complex scattering amplitude of the far field in Zitron and Karp [12]. The relation between the two amplitudes is then

$$f = \left(\frac{k}{2\pi}\right)^{\frac{1}{2}} e^{\frac{1}{4}\pi i} \tilde{f}.$$

4. Comparison with other expansions

This expansion is distinct from expansions obtained by other authors. A three-dimensional scalar expansion for large r is given by Sommerfeld [6]. A convergent three-dimensional vector expansion for large r was obtained by Wilcox [9]. A two-dimensional convergent expansion for large r was derived by Karp [3]. Expansions for large r which are more explicit have been obtained by Twersky [8] and by Burke, Censor and Twersky [1].

The present result, however, is given in terms of large d, rather than large r, thus providing an expansion in a new coordinate system whose origin O′ is distinct from the previous origin O. Here, d is a parameter. This expansion, therefore, may be regarded as an asymptotic form of an addition theorem. Thus, expansions for large d have an advantage over expansions for large r in that they allow a translation to a new origin.

The expansions for large d and large r may be contrasted as follows. The expansion for large d is good in a neighborhood of a predetermined point and is local in θ while the expansion for large r is good in an entire angular neighborhood of the original origin (i.e. global in θ). However, the expansion for large d is given in terms of wave functions, since derivatives of wave functions with respect to a parameter are wave functions, while the expansion for large r is not given in terms of wave functions.

5. Applications of the theorem

The expansion theorem may be employed in calculating the diffraction of an arbitrary two-dimensional radiating field. In such a calculation, the amplitude $f(\theta)$ for the incident radiation will be known. The incident radiation will then be represented in the form (6–7), in a coordinate system x, y fixed in the neighborhood of the scatterer. In addition, we must know the response amplitude $f_s(\theta, \theta_0)$ of the scatterer in direction θ, due to the excitation by an incident plane wave $\exp\{ik(x \cos \theta_0 + y \sin \theta_0)\} = v(\theta_0)$. Since θ_0 is a parameter, we can then calculate the response to any of the functions $v^{[\lambda]}(0)$ appearing in the expansion of the incident field. Since the response to a derivative of a plane wave is the derivative of the response to the plane wave, each term of the response to this incident radiation will have a coefficient identical to the one which appears in the expansion theorem (a linear combination of derivatives with respect to angle of incidence) operating on the response to a plane wave. For detailed illustrations, *cf.* Zitron and Karp [12], Zitron and Davis [10], Zitron and Davis [11], and Karp and Zitron [13].

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